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Dynamics of the Even-Binomial State in Some Quantum System

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The effect of the even-binomial state (the state which interpolates between the even-number state and the even-coherent state) on the Jaynes–Cummings model and resonance fluorescence for a single atom and for many cooperative atoms is discussed. The study of two quantum optical systems in such a state provides more insight into the continuous passage from the even-number state to the even-coherent state. The squeezing phenomenon and the correlation functions are examined.

1. INTRODUCTION

The most familiar state for the electromagnetic field is the Fock number state $|n\rangle$, which is an eigenstate of the photon number operator. Vigorous effort is being made to realize this state experimentally, for example, through atoms passing a quantized standing wave. Thus the momentum distribution of the outgoing atoms is sensitive to photon statistics for a small number of photons. Continual probing of the cavity by successive atoms results in the complete collapse of the field state to that of a number state (Holland *et al.*, 1991; Walls and Milburn, 1994).

A more appropriate basis for many optical applications is provided by the coherent states (Glauber, 1963). They are the closest quantum states to the classical description of the field realized in highly stabilized laser fields operating well above threshold (Sargent *et al.*, 1974). Another set of states

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consists of the squeezed states, which are minimum-uncertainty states (Yuen, 1976; Caves, 1981; Caves and Schumaker, 1985; Walls, 1986). These states have been generated experimentally through many schemes.[4]

There has been increased interest in studying combinations of these states. Some of these combinations interpolate between the Fock and coherent states. Such a state is the binomial state introduced by Stoler *et al.* (1985). A scheme for its production is given in Datoli *et al.* (1987). Another, introduced to bridge the thermal and coherent states, is the negative binomial state (Joshi and Lawande, 1989, 1991; Agarwal, 1992). The route from number to thermal states is provided by the generalized geometric state (Obada *et al.*, 1993; Batarfi *et al.*, 1995). Even- and odd-coherent states have been introduced to investigate the role of interference between coherent states (Gerry, 1993; Bužek and Knight, 1991; Bužek *et al.*, 1990, 1991; Hillery, 1987b; Vogel and Risken, 1990; Peřina, 1984; Sun *et al.*, 1991, 1992). Recently the even-binomial state has been introduced to interpolate between the even-number and even-coherent states (Abdalla *et al.*, 1994). It was reported that under suitable conditions, superposition of coherent states can be produced if a coherent state is allowed to propagate through an amplitude dispersive medium (Yurk and Stoler, 1986). Thus quantum superposition of binomial states can be produced in the same way, since the binomial state tends to a coherent state as a limiting case. Further, a method discussed recently (Gerry, 1992) to produce such a superposition through a Kerr medium in a Mach–Zehnder interferometer can be used to generate even-binomial states. The state that has been introduced takes the form

$$|\psi_e\rangle = \gamma \sum_{n=0}^{[M/2]} \binom{M}{2n}^{1/2} \eta^{2n}(\sqrt{1-|\eta|^2})^{(M-2n)}|2n\rangle \tag{1}$$

with $[M/2]$ being the largest integer less than or equal to $M/2$ and γ is a normalizing constant given by

$$|\gamma|^2 = 2[1 + (1 - 2|\eta|^2)^M]^{-1} \tag{1'}$$

In the limiting case when $\eta \to 0$ and $M \to \infty$ such that $M|\eta|^2 \to |\alpha|^2$, equation (1) reduces to

$$|\psi_e\rangle = (\operatorname{sech}|\alpha|^2)^{1/2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{2n!}} |2n\rangle \tag{2}$$

which represents the even-coherent state (Gerry, 1993; Bužek and Knight, 1991; Bužek *et al.*, 1990, 1992; Hillery, 1987b; Vogel and Risken, 1990; Peřina, 1984; Sun *et al.*, 1991, 1992).

[4]See the July 1987 issue of *Journal of Modern Optics*, Vol. 34, and the October 1987 issue of *Journal of the Optical Society of America B*.

In earlier work our concentration was on the statistical properties of the state, where we discussed in particular both the Glauber second-order correlation function and the squeezing phenomenon. We showed that for a certain value of the parameters M and η, antibunching and squeezing properties are exhibited. We also examined the quasi-probability distribution, W-Wigner, and Q-functions for this state, and discussed the connection to the even-coherent state. In the present work we extend the discussion to include some dynamical systems, namely the Jaynes and Cummings (1963) model and resonance fluorescence for a single atom and many cooperative atoms (Mollow, 1969), and we discuss the gradual behavior for the state of the radiation field when it changes from the even-number state to the even-coherent state. We point out that the Jaynes–Cummings model of a two-level atom interacting with a single mode of the electromagnetic field which describes the fundamental Bose–Fermi interaction is interesting not only from the theoretical point of view but also the experimental, since the availability of high-Q superconducting cavities at sub-Kelvin temperature makes it possible to realize it (Meschede *et al.*, 1985; Rempe *et al.*, 1987). In this case there is only one Rydberg atom interacting with one single mode of radiation, and the system in this case is known as a micromaser, which in general can be modeled by the Jaynes and Cummings (1963) model.

2. THE JAYNES–CUMMINGS MODEL

The simplest form of interaction between a two-level atom and a single quantized mode of the electromagentic field in the case of resonance is described by the Hamiltonian

$$H = \omega(a^\dagger a + \tfrac{1}{2}\sigma_z) + \lambda(a^\dagger\sigma_- + a\sigma_+) \tag{3}$$

where λ denotes the coupling between the atom and the field, and a and $a^\dagger$ are the boson annihilation and creation operators of the field, with $[a, a^\dagger] = 1$; σ_z, σ_-, and σ_+ are the atomic pseudo-spin operators, with $[\sigma_+, \sigma_-] = \sigma_z$ and $[\sigma_\pm, \sigma_z] = \mp 2\sigma_\pm$.

Now suppose we prepare the atom to be initially in the excited state $|e\rangle$ and the field to be initially in the even binomial state; then we find the initial atom-field state is the product of the atomic superposition states given by

$$|\psi(0)\rangle = \gamma \sum_{n=0}^{[M/2]} B_{2n}^M |e, 2n\rangle \tag{4}$$

where B_{2n}^M is the distribution of the photons defined as

$$B_{2n}^M = \binom{M}{2n}^{1/2} \eta^{2n}(\sqrt{1 - |\eta|^2})^{(M-2n)} \tag{4'}$$

For $t > 0$ the state $|\psi(t)\rangle$ in the interaction picture may be obtained from the Hamiltonian (3) in the form

$$|\psi(t)\rangle = \gamma \sum_{n=0}^{[M/2]} B_{2n}^{M}[\cos \tau\sqrt{2n + 1}\,|2n, e\rangle - i \sin \tau\sqrt{2n + 1}\,|2n + 1, g\rangle] \tag{5}$$

where $\tau = \lambda t$. From equation (5) we can easily obtain the expectation value for any atomic or field operator. For example, we can calculate the temporal evolution for $(a^{\dagger})^{2s}$ to give

$$\begin{aligned}\langle a^{\dagger 2s}\rangle = |\gamma|^2\eta^{*2s} \sum_{n=0}^{[M/2]} \binom{M}{2n}\left[\frac{(M - 2n)!}{(M - 2n - 2s)!}\right]^{1/2} \eta^{4n} \\ \times (1 - |\eta|^2)^{(M-2n)}\Bigg[\cos \tau\sqrt{2n + 2s + 1} \cos \tau\sqrt{2n + 1} \\ + \left(\frac{2n + 2s + 1}{2n + 1}\right)^{1/2} \\ \times \sin \tau\sqrt{2n + 2s + 1} \sin \tau\sqrt{2n + 1}\Bigg]\end{aligned} \tag{6}$$

and the calculation of the expectation value of the photon number $\hat{N} = a^{\dagger}a$ gives

$$\langle \hat{N}\rangle = |\gamma|^2 \sum_{n=0}^{[M/2]} |B_{2n}^{M}|^2(2n + \sin^2\tau\sqrt{2n + 1}) \tag{6'}$$

Note that the expectation value of any odd power as well as any power higher than M for the operators a and $a^{\dagger}$ vanishes due to the nature of the even binomial state. The nonclassical effects of the above system will be considered in the following section.

3. NONCLASSICAL EFFECTS

In this section we use the results obtained in the previous section to discuss higher order squeezing phenomena (amplitude-squared squeezing) as well as the correlation function.

3.1. Amplitude-Squared Squeezing

The phenomenon of squeezing is distinguished by the property that the quantum fluctuations in one of the field quadratures is smaller than those associated with the coherent states of light. In fact this phenomenon has been extended to higher order squeezing. The concept of higher order squeezing has been discussed by many authors (Hong and Mandel, 1985a,b) and in this subsection we discuss the production of higher order squeezing in the sense of Hillery's (1987a,b) definition. This type of squeezing is known as amplitude-squared squeezing, and arises in a natural way in second-harmonic generation and in a number of nonlinear optical processes. This phenomenon is defined through the fluctuations in the operators

$$d_1 = \frac{1}{2} a^{\dagger 2} + a^2) \tag{7a}$$

$$d_2 = \frac{i}{2} (a^{\dagger 2} - a^2) \tag{7b}$$

These operators satisfy the commutation relation

$$[d_1, d_2] = i[1 + 2\hat{N}] \tag{8}$$

The field is said to be in amplitude-squared squeezed state if

$$\Delta^2 d_1 \quad \text{or} \quad \Delta^2 d_2 < \tfrac{1}{2} + \langle \hat{N} \rangle \tag{9}$$

Equation (9) can be rewritten in terms of a and $a^\dagger$ as follows (Mahran and Obada, 1989):

$$S_1(t) = \tfrac{1}{4}[2\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle + (\langle a^{\dagger 4} + a^4 \rangle) - \langle (a^{\dagger 2} + a^2)^2 \rangle] < 0 \tag{10a}$$

and

$$S_2(t) = \tfrac{1}{4}[2\langle \hat{N}^2 \rangle + 2\langle \hat{N} \rangle + (\langle a^{\dagger 4} \rangle + \langle a^4 \rangle) - \langle (a^{\dagger 2} - a^2)^2 \rangle] < 0 \tag{10b}$$

where

$$\langle \hat{N}^2 \rangle = |\gamma|^2 \sum_{n=0}^{[M/2]} |B_{2n}^M|^2 [4n^2 + (4n + 1) \sin^2 \tau \sqrt{2n + 1}] \tag{10c}$$

In Fig. 1 we plot the amplitude-squared squeezing against the scaled time τ. At time $\tau > 0$ we observe that nonclassical negative values appear in the X component of equation (10) which, corresponding to S_1 or S_2, depends upon the phase of the parameter η. From Fig. 1, for a fixed mean photon number $\bar{n} = 5$, it is easy to realize in general that for short and long periods of time the amplitude-squared squeezing is pronounced, but this depends on the value of the parameters η and M. We have illustrated three different cases. The first one is $M = 10$ and $\eta = 0.7$, and the other two cases are $M = 20$ and $\eta = 0.5$, and $M = 30$ and $\eta = 0.4$. The observed amount of amplitude-squared squeezing for both short and long time intervals is more pronounced

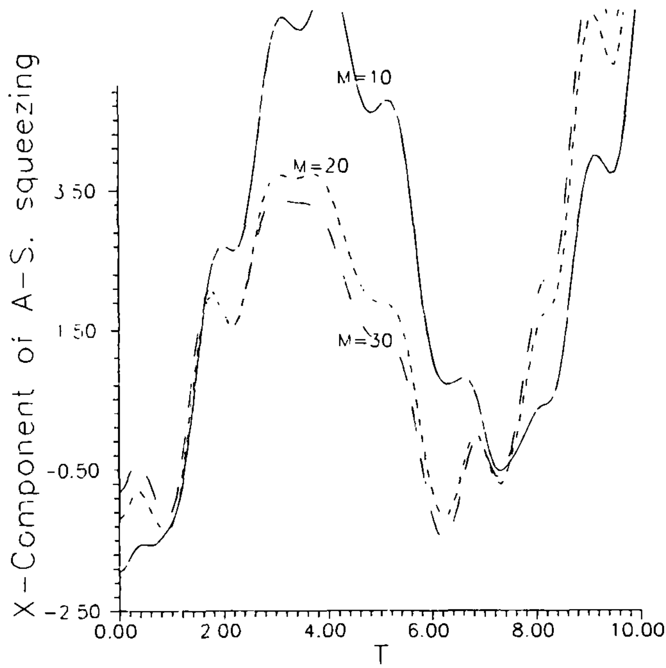

Fig. 1. The amplitude-squared squeezing equation (10a) versus the time for $\bar{n} = 5$ and different values of M, η.

for the case $M = 10$, $\eta = 0.7$ than that for the other two cases. We also observe that, as the value of the parameter M increases and the parameter η decreases, the system approaches the even-coherent-state behavior and this can be noticed for the case $M = 30$, $\eta = 0.4$. Finally, as stated in Abdalla *et al.* (1994), normal squeezing is observed at $t = 0$; however, in the present case at $t > 0$ the normal squeezing is washed out. This is due to the highly noisy character of the state produced during the interaction. We also point out that at $t = 0$ the observed amount of amplitude-squared squeezing is much larger than that observed for $t > 0$ in the present case, and this also is due to the noise that appeared during the course of interaction. Furthermore, our numerical investigation for the case when the initial mean photon number $\bar{n} = 2$ with $M = 30$ and $\eta = 0.3$ shows that no normal squeezing can be observed, while there is a small amount of amplitude-squared squeezing for both short and long time intervals.

3.2. Sub-Poissonian Distributions

We turn our attention to the Glauber second-order correlation function $g^{(2)}(t)$, which is defined as

$$g^{(2)}(t) = \frac{\langle \hat{N}^2(t) \rangle - \langle \hat{N}(t) \rangle}{\langle \hat{N}(t) \rangle^2} \tag{11}$$

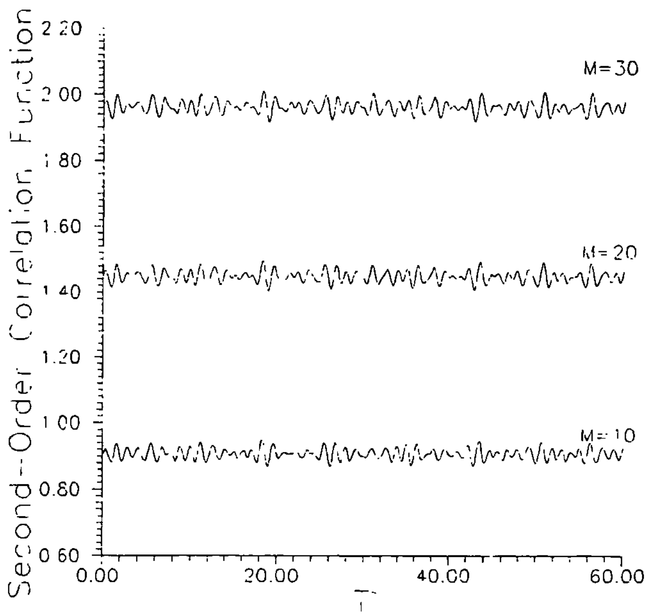

Fig. 2. The second-order correlation function $g^{(2)}(t)[(g^{(2)}(t) + 0.5]$ for $M = 20$ and $[g^{(2)}(t) + 1.0]$ for $M = 30$ versus the time for the same values of $\bar{n}$ as in Fig. 1.

The field is said to be sub-Poissonian if the correlation function $g^{(2)}(t)$ is less than unity and super-Poissonian if $g^{(2)}(t) > 1$. In Fig. 2 we plot the time evolution of the function $g^{(2)}(t) > 1$ against the time τ with the mean photon number $\bar{n} = 5$ and for different values of M and η. We notice that the function $g^{(2)}(t)$ in general shows oscillation behavior. Accordingly, as we increase the value of M, the correlation functions starts to oscillate between sub-Poissonian and super-Poissonian behavior (see, for example, $M = 30$). On the other hand, the distribution function starts to be super-Poissonian for small values of η and large values of M. This of course is due to the fact that the even binomial state will tend to the even-coherent state as $\eta \to 0$ and $M \to \infty$. Finally, we observed in both cases $M = 10$ and $M = 20$ that the state is antibunched. However, we can see more antibunched for lower M and this indicates that the system approaches the even-number state as the value of the parameter is decreasing. For the case of $M = 30$ the state is oscillating between bunching and antibunching.

4. RESONANCE FLUORESCENCE

This phenomenon concerns a radiatively decaying two-level atomic system coupled to an external radiation field in free space. The steady-state regime (i.e., $t \to \infty$) is considered here. We consider especially the cases of a single atom ($N = 1$) and the thermodynamic limit in the cooperative many-atom system in which $N \to \infty$.

4.1. Single Atom

When the external field is described in a single-mode-number state, we find in the steady state that the mean atomic inversion for a single two-level atom is given by (Hassan *et al.*, 1990)

$$\langle S_z(\infty)\rangle n = -\frac{1}{2}\sum_{m=0}^{n}(-)^m \frac{n!}{(n-m)!}b^{2m} = -(b^2)^n L_n^{(-n-1)}(-b^{-2}) \quad (12)$$

where $b^2 = 2\hbar^{-2}g^2/(\delta^2 + \frac{1}{4}\Gamma^2)$; g is the coupling constant and Γ is the A-coefficient; δ is the frequency detuning between the atomic transition frequency and that of the field, and $L_n^{(a)}(x)$ is the generalized Laguerre polynomial.

From the even-binomial state defined by equation (1), we get the following expression for the mean atomic inversion:

$$\langle S_z(\infty)\rangle_{EB} = \sum_{n=0}^{[M/2]} P_{2n}(\eta, M)\langle S_z(\infty)\rangle_{2n} \quad (13)$$

where

$$P_{2n}(\eta, M) = |\langle 2n|\psi_e\rangle|^2$$
$$= |\gamma|^2 \frac{M!}{2n!(M-2n)!}\left[\frac{|\eta^2|}{1-|\eta|^2}\right]^{2n}(1-|\eta|^2)^M \quad (14)$$

The result (13) can be rearranged and summed to take the form

$$\langle S_z(\infty)\rangle_{EB} = -\frac{|\gamma|^2}{2}[(b^2|\eta|^2)^M L_M^{-M-1}(-b^2|\eta|^2)$$
$$+ (-b^2|\eta|^2)^M L_M^{-(M-1)}(b^2|\eta|^2)] \quad (15)$$

This formula represents the mean atomic inversion for the even-binomial state. Note that for $\eta = 0$ we get the vacuum state and as $M \to \infty$ and $\eta \to 0$ such that $M|\eta|^2 \to |\alpha|^2$, we get the result for the even-coherent state, which is equal to

$$\langle S_z(\infty)\rangle_{EC} = -\frac{1}{4}\operatorname{sech}|\alpha|^2[(1 + b^2|\alpha|^2)^{-1} + (1 - |\alpha|^2 b^2)^{-1}] \quad (15')$$

4.2. The Thermodynamic Limit ($N \to \infty$)

In the case of the N-cooperative-atom resonance fluorescence and in the limit $N \to \infty$, the scaled atomic inversion at exact resonance in the number state field is given by

$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_n}{N}\right] = -\frac{1}{2}C_n\left(\frac{1}{2}; X^2\right)$$

$$= -\frac{1}{2}\sum_{s=0}^{n}\binom{1/2}{s}\begin{bmatrix} n \\ s \end{bmatrix} s!\,(-X^{-2})^s \tag{16}$$

where $X^2 = \Gamma N/(2\hbar^{-2}g^2)$ and C_n are the Poisson–Charlier polynomials. For the even-binomial state, we get the expression

$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_{EB}}{N}\right] = -\frac{1}{2}\sum_{n=0}^{[M/2]} P_{2n}(\eta, M)C_{2n}\left(\frac{1}{2}; X^2\right) \tag{17}$$

which can be rearranged and summed in the form

$$\lim_{N\to\infty}\left[\frac{\langle S_z(\infty)\rangle_{EB}}{N}\right] = -\frac{\gamma^2}{4}\left[C_M\left(\frac{1}{2}, X^2|\eta|^2\right) + C_M\left(\frac{1}{2}, -X^2|\eta|^2\right)\right] \tag{18}$$

In Figs. 3a and 3b we plot equations (15) and (18) against $|\eta|$ for different values of M. We note that as we increase the value of M, the curve approaches saturation faster. However, the rate in the thermodynamic limit is slower than for the case of a single atom.

5. CONCLUSION

We have examined the Jaynes–Cummings model as well resonance fluorescence against the even-binomial state (the state which interpolates between the even-number state and the even-coherent state).

For the Jaynes–Cummings model, which shows dynamical behavior, we discussed amplitude-squared squeezing, and showed that the squeezing is more pronounced at the time $t = 0$ than for the time $t > 0$. We also extended our discussion to include the sub-Poissonian distributions, where we examined the second-order correlation function $g^{(2)}(t)$ against the time τ. We found that the system shows bunching and antibunching behavior, but this depends upon the value of both M and η. Finally, we considered the resonance fluorescence for the single-atom and the thermodynamic limit and we showed that the rate of change in the case of the thermodynamic limit is slower than the rate of change in the single-atom case. To complete our discussion we plot the inversion $\langle\hat{\sigma}_z(t)\rangle$ for the present model against the time τ. We notice that as the time τ increases (see Fig. 4), the model shows pronounced collapse-and-revival behavior associated with Rabi oscillations. Also, as many different components in the summation get out of phase, collapse behavior of the inversion is possible. On the other hand, it is well

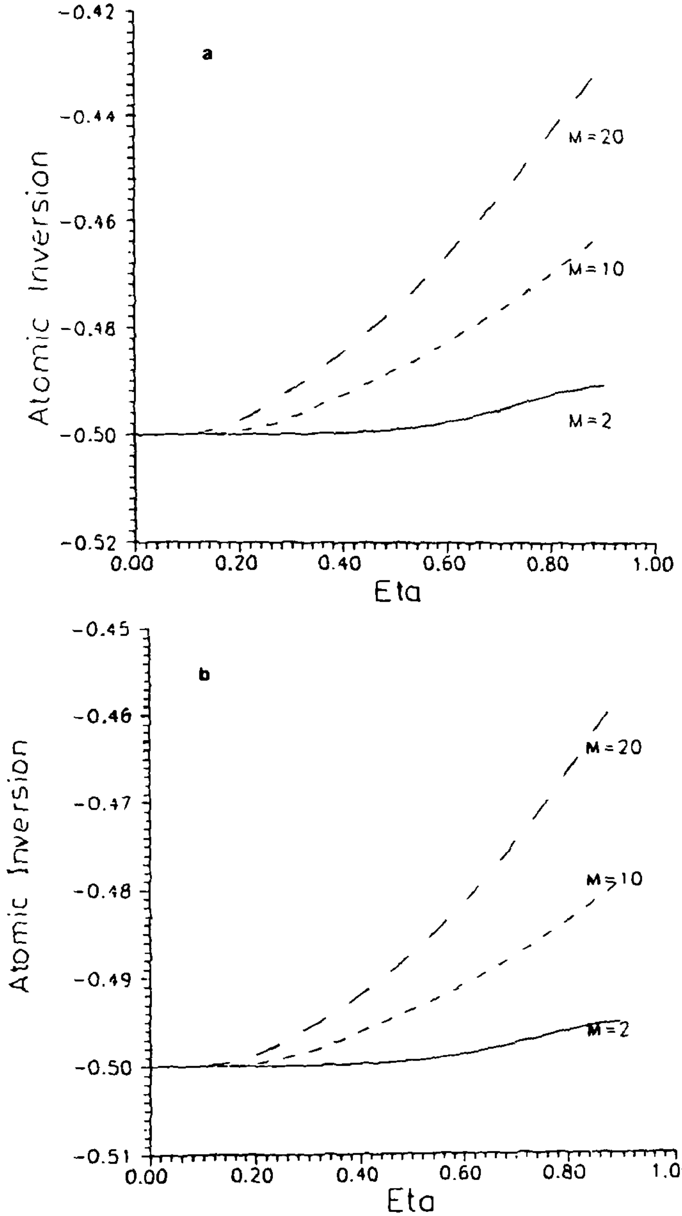

Fig. 3. (a) The atomic inversion equation (15) versus the parameter η with $M = 2$, $M = 10$, and $M = 20$. (b) The atomic inversion equation (18) versus the parameter η with $M = 2$, $M = 10$, and $M = 20$.

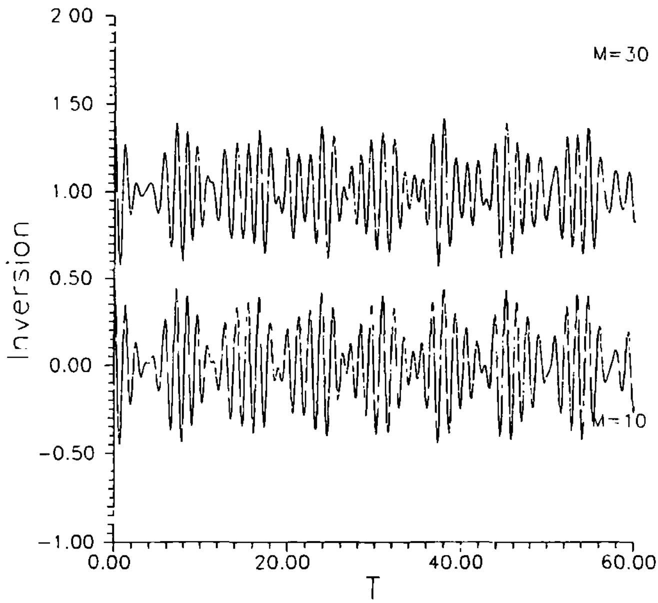

Fig. 4. The atomic inversion of $(\langle\hat{\sigma}_z(t)\rangle + 1.0)$ for the Jaynes–Cummings model versus the time for $M = 30$ and $M = 10$ with $\bar{n} = 5$.

known that the revivals are a manifestation of the quantum nature of the interacting mode, which is mathematically reflected in the discrete summation (Eberly *et al.*, 1980; Narozhny *et al.*, 1981; Yoo *et al.*, 1981).

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